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ON FØLNER NETS AND CROSSED PRODUCTS

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ABSTRACT. Let $\mathcal{M}$ be a von Neumann algebra that has a Følner net . In the present article we give conditions that guarantee that the von Neumann crossed product of $\mathcal{M}$ with an amenable discrete group has a Følner net . The Følner net for the crossed product is given explicitly and the result is applied to the rotation algebra.

CONTENTS

1. INTRODUCTION

Many notions in group theory have their counterparts in the context of operator algebras. Følner nets were introduced in the context of operator algebras by Alain Connes in his seminal paper [20, Section V] (see also [21, 31, 32]). This notion is an algebraic analogue of Følner's characterization of amenable discrete groups (see Section 2 for precise definitions) and was used by Connes as an essential tool in the classification of injective type II_1 factors. Recently, this circle of ideals has been used to define a new invariant for a general separable type II_1 factor that measures how badly the factor fails to satisfy Connes' Følner type condition.

In addition to these theoretical developments, Følner nets have been used in spectral approximation problems: given a sequence of linear operators $\{T_n\}_{n\in\mathbb{N}}$ in a complex Hilbert space $\mathcal{H}$ that approximates an operator T in a suitable sense, a natural question is how do the spectral objects of T relate with those of T_n as n grows (for general references see, e.g., [19, 4]). This line of research, emphasizing now the importance of the operator algebraic point of view, was initiated by Arveson in a series of papers [1, 2, 3] (see also [11]). Among other results, Arveson gave conditions that guarantee that the essential spectrum of a selfadjoint operator T may be recovered from the sequence of eigenvalues of certain finite dimensional compressions T_n. These results were then extended by Bédos who systematically applied the concept of Følner net to spectral approximation problems [8, 7, 6] (see also [18]).

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Let $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ be a set of bounded linear operators on the complex separable Hilbert space $\mathcal{H}$. A net of non-zero finite rank orthogonal projections $\{P_i\}_{i\in I}$ is called a Følner net for $\mathcal{T}$, if

$$\lim_i \frac{\|TP_i - P_iT\|_2}{\|P_i\|_2} = 0, \quad T \in \mathcal{T} ,$$

where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm. The existence of a Følner net for a set of operators $\mathcal{T}$ is a weaker notion than quasidiagonality. Recall that a set of operators $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ is said to be quasidiagonal if there exists a net of finite-rank projections $\{P_i\}_i$ as before such that

$$\lim_i \|TP_i - P_iT\| = 0 , \quad T \in \mathcal{T} .$$

It is easy to show that if $\{P_i\}_{i\in I}$ quasidiagonalizes the set of operators $\mathcal{T}$, then it is also a Følner net for $\mathcal{T}$. Moreover, the Følner condition above can be understood as a quasidiagonality condition, but relative to the growth of the dimension of the underlying spaces. Quasidiagonality has also deep consequences for the structure of C*-algebras (see, e.g., Chapter 7 in [17] or [15, 9, 14, 39]) and is very useful notion in spectral approximation problems (cf. [16, 24]).

The question when a crossed product of a quasidiagonal C*-algebra is again quasidiagonal has been addressed several times in the past (see, e.g., Section 11 in [15]). In Lemma 3.6 of [30] it is shown that if $\mathcal{A}$ is a unital separable quasidiagonal C*-algebra with almost periodic group action $\alpha\colon \mathbb{Z} \to \mathrm{Aut}\mathcal{A}$, then the C*-crossed-product $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ is again quasidiagonal. This result has been extended recently by Orfanos to C*-crossed products $\mathcal{A} \rtimes_\alpha \Gamma$, where now Γ is a discrete, countable, amenable, residually finite group (cf. [26]). In this more general case there is again a condition on the group action of Γ on $\mathcal{A}$. This result has been applied to generalized Bunce-Deddens algebras in [27].

The aim of the present paper is to give conditions and construct explicitly Følner nets for the cross-product of the von Neumann algebra $\mathcal{M}$ that has a Følner net and a discrete amenable group Γ with Følner net Γ_i for Γ (see Theorem 3.3 for a precise statement). In our case we have to assume also a certain compatibility condtion between the choice of the Følner net for $\mathcal{M}$ and the Følner net $\{\Gamma_i\}_i$ for Γ through the action $\alpha\colon \Gamma \to \mathrm{Aut}\mathcal{M}$. In the existing literature there are only a few concrete constructions of Følner nets in the context of von Neumann algebras. In general, it is not true that a Følner net for a concrete C*-algebra is a Følner net for its weak closure. Bédos shows in Proposition 4 of [6] that there is a Følner net for the twisted group von Neumann algebra of a discrete amenable group Γ which is canonically associated the Følner net $\{\Gamma_i\}_i$ for Γ (cf. Theorem 3.2). Boca defines in Proposition 1.15 of [10] a Følner sequence for the von Neumann algebra $\mathcal{N}_\theta$ generated by the rotation algebra $\mathcal{A}_\theta$ in a tracial state. The choice of this sequence is adapted to the structure of the corresponding GNS Hilbert space. Our main result may also be applied to the rotation algebra $\mathcal{A}_\theta$ since it can be interpreted as a crossed product $\mathcal{A}_\theta \cong C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$, where $C(\mathbb{T})$ are the continuous function on the unit circle and the action of $\mathbb{Z}$ is given by a suitable rotation of the argument (see Subsection 3.2 for details). In particular, we will specify a different Følner net for the von Neumann crossed product $\mathcal{N}_\theta \cong L^\infty(\mathbb{T}) \otimes_\alpha \mathbb{Z}$ than the one considered by Boca. From the point of view of spectral approximation problems it is more natural to consider the von Neumann algebra point of view since these alegebras contain all the spectral projections of the normal operators contained in it. In fact, the Følner nets considered for the rotation algebra may be aplied to study spectral approximation problems of very interesting operators like periodic magnetic Schrödinger operators or almost Mathieu operators which are contained in $\mathcal{N}_\theta$ (cf. [36, 10]).

2. FØLNER NETS AND QUASIDIAGONALITY

A discrete group Γ is amenable if it has an invariant mean, i.e. there is a state ψ on the von Neumann algebra $\ell^\infty(\Gamma)$ such that

$$\psi(u_\gamma g) = \psi(g) , \quad \gamma \in \Gamma , \quad g \in \ell^\infty(\Gamma) ,$$

where u is the left-regular representation on $\ell^2(\Gamma)$. A Følner net for Γ is a net of non-empty finite subsets $\Gamma_i \subset \Gamma$ that satisfy

$$\lim_i \frac{|(\gamma\Gamma_i)\triangle\Gamma_i|}{|\Gamma_i|} = 0 \qquad \text{for all} \quad \gamma \in \Gamma\,, \tag{2.1}$$

where $\triangle$ denotes the symmetric difference and $|\Gamma_i|$ is the cardinality of Γ_i. Then, Γ has a Følner net if and only if Γ is amenable (cf. Chapter 4 in [29]). Some authors require, in addition to Eq. (2.1), that the net is increasing and complete, i.e. $\Gamma_i \subset \Gamma_j$ if $i \preccurlyeq j$ and $\Gamma = \cup_i \Gamma_i$. We will not make these additional assumptions here.

The algebraic counterpart of this notion was introduced by Connes in his seminal paper [20] as an important tool in the analysis of the injective type II_1 factor (see also [21, Section 2]). Later this notion was used in a systematic way by Bédos in the context of eigenvalue distribution problems (cf. [7, 6]).

Definition 2.1. *Let $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ be a set of bounded linear operators on the complex separable Hilbert space $\mathcal{H}$. A net of non-zero finite rank orthogonal projections $\{P_i\}_{i\in I}$ is called a Følner net for $\mathcal{T}$, if one of the four following equivalent conditions holds for all $T \in \mathcal{T}$:*

$$\lim_i \frac{\|TP_i - P_iT\|_p}{\|P_i\|_p} = 0, \qquad p \in \{1,2\} \tag{2.2}$$

or

$$\lim_i \frac{\|(I-P_i)TP_i\|_p}{\|P_i\|_p} = 0, \qquad p \in \{1,2\}\,, \tag{2.3}$$

where $\|\cdot\|_1$ and $\|\cdot\|_2$ are the trace-class and Hilbert-Schmidt norms, respectively.

The equivalence of the conditions in this definition is proven in Lemma 1 of [6]. An immediate consequence of the previous definition is that the existence of a Følner net for a C*-algebra can be already verified on a generating set.

Proposition 2.2. *Let $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ be a set of operators. If $\{P_i\}_i$ is a Følner net for $\mathcal{T}$, then it is also a Følner net for the C*-algebra C*$(\mathcal{T})$ generated by $\mathcal{T}$.*

The existence of a Følner net for a set of operators $\mathcal{T}$ is a weaker notion than quasidiagonality. Recall that a set of operators $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ is said to be quasidiagonal if there exists a net of finite-rank projections $\{P_i\}_{i\in I}$ as in Definition 2.1 such that

$$\lim_i \|TP_i - P_iT\| = 0\,, \quad \text{for all } T \in \mathcal{T}\,. \tag{2.4}$$

The Følner condition, in particular Eq. (2.2), can be understood as a quasidiagonality condition, but relative to the growth of the dimension of the underlying spaces. It can be easily shown that if $\{P_i\}_i$ quasidiagonalizes a family of operators $\mathcal{T}$, then this net of non-zero finite rank orthogonal projections is also a Følner net for $\mathcal{T}$. The notion of quasidiagonality can be extended to the abstract C*-setting, where one must be particularly careful in the non-separable case (cf. [15, 17]).

The existence of a Følner net has important structural consequences. For the next result we need to recall the following notion: a hypertrace for a C*-algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ is a state Ψ on $B(\mathcal{H})$ that is centralised by $\mathcal{A}$, i.e.

$$\Psi(XA) = \Psi(AX)\,, \quad X \in B(\mathcal{H})\,,\ A \in \mathcal{A}\,.$$

Hypertraces are the algebraic analogue of the invariant mean mentioned above.

Proposition 2.3. *Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a concrete C*-algebra. Then $\mathcal{A}$ has a Følner net iff $\mathcal{A}$ has a hypertrace.*

To simplify expressions in the rest of the article we introduce the notion of the commutator of two operators: $[A,B] := AB - BA$.

Examples:

(i) The unilateral shift is a prototype that shows the difference between the notions of Følner sequences and quasidiagonality. On the one hand, it is a well-known fact that the unilateral unilateral shift S is not a quasi-diagonal operator. (This was shown by Halmos in [22]; in fact, in this reference it is shown that S is not even quasi-triangular.) In the setting of abstract C*-algebras it can also be shown that a C*-algebra containing a proper (i.e. non-unitary) isometry is not quasi-diagonal (see, e.g. [15, 17]). It can be shown, though, that certain weighted shifts are quasidiagonal (cf. [33]).

On the other hand, it is easy to give a Følner sequence for S. In fact, define S on $\mathcal{H} := \ell^2(\mathbb{N}_0)$ by $Se_i := e_{i+1}$, where $\{e_i \mid i = 0, 1, 2, \dots\}$ is the canonical basis of $\mathcal{H}$ and consider for any n the orthogonal projections P_n onto $\mathrm{span}\{e_i \mid i = 0, 1, 2, \dots, n\}$. Then

$$\big\|[P_n, S]\big\|_2^2 = \sum_{i=1}^{\infty} \Big\|[P_n, S]e_i\Big\|^2 = \|e_{n+1}\|^2 = 1$$

and

$$\frac{\big\|[P_n, S]\big\|_2}{\|P_n\|_2} = \frac{1}{\sqrt{n}} \underset{n\to\infty}{\longrightarrow} 0 \ .$$

A similar argument shows directly that $\{P_n\}_n$ is a Følner sequence for any power S^k, $k \in \mathbb{N}$.

(ii) We will verify that the previous sequence is not a Følner sequence for the following realization of the isometries generating the Cuntz algebra $\mathcal{O}_2$. Consider on $\mathcal{H} := \ell^2(\mathbb{N}_0)$ the isometries $S_1 e_i := e_{2i}$ and $S_2 e_i := e_{2i+1}$ which satisfy $S_i^* S_j = \delta_{ij}$, $i, j \in \{1, 2\}$, and $S_1 S_1^* + S_2 S_2^* = \mathbb{1}$. In this case we have

$$\big\|[P_{2n}, S_1]\big\|_2^2 = \sum_{i=1}^{\infty} \Big\|[P_{2n}, S_1]e_i\Big\|^2 = \sum_{i=n+1}^{2n} \|e_{2i}\|^2 = n\,.$$

and the limit in (2.2) does not exist. (Similar arguments for S_2.)

(iii) Let $T \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator. If a sequence of non-zero finite rank orthogonal operators $\{P_n\}_n$ satisfies

$$\sup_{n\in\mathbb{N}} \|(\mathbb{1} - P_n)TP_n\|_2 < \infty \ ,$$

then $\{P_n\}_n$ is clearly a Følner sequence for T. Concrete examples satisfying the preceding condition are self-adjoint operators with a band-limited matrix representation (see, e.g., [2, 3]). Band limited operators together with quasidiagonal operators are the essential ingredients in the solution of Herrero's approximation problem, i.e. the characterization of the closure of block diagonal operators with bounded blocks (see Chapter 16 in [17] for a comprehensive presentation).

3. FØLNER NETS AND CROSSED PRODUCTS

The crossed product may be seen as a new von Neumann algebra constructed from a given von Neumann algebra which carries a certain group action. This procedure goes back to the pioneering work of Murray and von Neumann on rings of operators. Standard references which present the crossed product construction with small variations are [35, Chapter 4], [37, Section V.7] or [25, Section 8.6 and Chapter 13].

Throughout this section $\mathcal{M}$ denotes a concrete von Neumann algebra acting on a complex separable Hilbert space $\mathcal{H}$. We shall assume that α is an automorphic representation of a countable discrete group Γ on $\mathcal{M}$, i.e.

$$\alpha \colon \Gamma \to \mathrm{Aut}\,\mathcal{M} \ .$$

The crossed product is a new von Neumann algebra constructed with the previous ingredients and that lives on the separable Hilbert space

$$\mathcal{K} := \ell^2(\Gamma) \otimes \mathcal{H} \cong \bigoplus_{\gamma\in\Gamma} \mathcal{H}_\gamma \tag{3.1}$$

where $\mathcal{H}_\gamma \equiv \mathcal{H}$ for all $\gamma \in \Gamma$. To make this notion precise we introduce representations of $\mathcal{M}$ and Γ on $\mathcal{K}$, respectively: for $\xi \in \mathcal{K}$ we define

$$(\pi(M)\xi)_\gamma := \alpha_\gamma^{-1}(M)\,\xi_\gamma\,, \quad M \in \mathcal{M}, \tag{3.2}$$

$$(U(\gamma_0)\,\xi)_\gamma := \xi_{\gamma_0^{-1}\gamma}\,. \tag{3.3}$$

The crossed product of $\mathcal{M}$ by Γ is the von Neumann algebra in $\mathcal{K}$ generated by these operators, i.e.,

$$\mathcal{N} = \mathcal{M} \otimes_\alpha \Gamma := \Big(\{\pi(M) \mid M \in \mathcal{M}\} \cup \{U(\gamma) \mid \gamma \in \Gamma\} \Big)'' \subset \mathcal{L}(\mathcal{K})\,,$$

where the prime denotes the commutant in $\mathcal{L}(\mathcal{K})$ and $\mathcal{A}'' := (\mathcal{A}')'$ for any $\mathcal{A} \subset \mathcal{L}(\mathcal{K})$. A characteristic relation for the crossed product is

$$\pi\left(\alpha_\gamma(M)\right) = U(\gamma)\pi(M)U(\gamma)^{-1}\,, \tag{3.4}$$

that is, π is a covariant representation of the W^*-dynamical system $(\mathcal{M}, \Gamma, \alpha)$.

Remark 3.1. *Later we will need the following characterization of the elements in the crossed product: consider the identification* $\mathcal{K} \cong \bigoplus_{\gamma\in\Gamma} \mathcal{H}_\gamma$ *with* $\mathcal{H}_\gamma \equiv \mathcal{H}$, $\gamma \in \Gamma$. *Then, every* $T \in \mathcal{L}(\mathcal{K})$ *can be written as an infinite matrix* $(T_{\gamma'\gamma})_{\gamma',\gamma\in\Gamma}$ *with entries* $T_{\gamma'\gamma} \in \mathcal{L}(\mathcal{H})$. *Any element* N *in the crossed product* $\mathcal{N} \subset \mathcal{L}(\mathcal{K})$ *has the form*

$$N_{\gamma'\gamma} = \alpha_\gamma^{-1}\left(A(\gamma'\gamma^{-1})\right)\,, \quad \gamma'\,, \gamma \in \Gamma\,, \tag{3.5}$$

for some mapping $A\colon \Gamma \to \mathcal{M} \subset \mathcal{L}(\mathcal{H})$. *Since any* $N \in \mathcal{N}$ *is bounded operator it follows that*

$$\sum_{\gamma\in\Gamma} \|A(\gamma)\|^2 < \infty\,. \tag{3.6}$$

For example, the matrix form of the product of generators $N := \pi(M) \cdot U(\gamma_0)$, $M \in \mathcal{M}$, $\gamma_0 \in \Gamma$ *is given by*

$$N_{\gamma'\gamma} = \alpha_{\gamma'}^{-1}(M)\,\delta_{\gamma',\gamma_0\gamma} = \alpha_{\gamma'}^{-1}\left(A(\gamma'\gamma^{-1})\right)\,, \text{where } A(\widetilde{\gamma}) := \begin{cases} M \text{ if } \widetilde{\gamma} = \gamma_0 \\ 0 \quad \text{otherwise} \end{cases}\,. \tag{3.7}$$

This implies that any function $A\colon \Gamma \to \mathcal{M}$ *with finite support determines by means of Eq. (3.5) an element in the crossed product.*

3.1. **Construction of a Følner nets.** The aim of the present section is to give a canonical example of a Følner net for the crossed product von Neumann algebra $\mathcal{N}$ constructed above. Since $\mathcal{N} \subset \mathcal{L}(\mathcal{K})$ with $\mathcal{K} = \ell^2(\Gamma) \otimes \mathcal{H}$, our net is canonical in the sense that it uses explicitly a Følner net for Γ and a net of projections on $\mathcal{H}$ (cf. Theorem 3.3). We will assume first that the von Neumann algebra $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ has a Følner net $\{Q_i\}_{i\in I}$.

We begin recalling Proposition 4 of [6]:

Theorem 3.2. *Assume that the group* Γ *is amenable and denote by* $\{P_i\}_i$ *the net of orthogonal finite-rank projections from* $\ell^2(\Gamma)$ *onto* $\ell^2(\Gamma_i)$ *associated to a Følner net* $\{\Gamma_i\}_i$ *for the group* Γ *(cf. Section 2). Then* $\{P_i\}_i$ *is a Følner net for the group von Neumann algebra*

$$\mathcal{N}_\Gamma := \{\overline{U}(\gamma) \mid \gamma \in \Gamma\}'' \subset \mathcal{L}(\ell^2(\Gamma))\,,$$

where $\overline{U}$ *is the left regular representation of* Γ *on* $\ell^2(\Gamma)$.

Note that the previous result means that the net $\{P_i\}_i$ satisfies the four equivalent conditions in Definiton 2.1 for any element in the group von Neumann algebra $\mathcal{N}_\Gamma$.

Theorem 3.3. *Consider the previous nets $\{P_i\}_i$ and $\{Q_i\}_i$ of finite rank orthogonal projections defined on $\ell^2(\Gamma)$ and $\mathcal{H}$, respectively. Assume that the action of Γ on $\mathcal{M}$ satisfies:*

$$\lim_i \left(\max_{\gamma\in\Gamma_i} \frac{\| \left[Q_i, \alpha_\gamma^{-1}(M)\right] \|_2}{\|Q_i\|_2} \right) = 0 \,, \quad \text{for all} \;\; M \in \mathcal{M} \,. \tag{3.8}$$

Then the net $\{R_i\}_i$ with

$$R_i := P_i \otimes Q_i$$

is a Følner net for the crossed product $\mathcal{N} = \mathcal{M} \otimes_\alpha \Gamma$, i.e. the four equivalent conditions in Definition 2.1 are satisfied.

Proof. Step 1: we consider the indentification $\mathcal{K} \cong \underset{\gamma\in\Gamma}{\oplus} \mathcal{H}_\gamma$, $\mathcal{H}_\gamma \equiv \mathcal{H}$. In this case any element N in the crossed product $\mathcal{N} \subset \mathcal{L}(\mathcal{K})$ can be seen as an infinite matrix of the form

$$N_{\gamma'\gamma} = \alpha_\gamma^{-1}\left(A(\gamma'\gamma^{-1})\right) \;, \quad \gamma'\,, \gamma \in \Gamma \,, \tag{3.9}$$

where $\gamma \mapsto A(\gamma)$ is a mapping from $\Gamma \to \mathcal{M}$ (cf. Remark 3.1). Moreover, defining the unitary map

$$W\colon \mathcal{H} \otimes \ell^2(\Gamma) \to \underset{\gamma\in\Gamma}{\oplus} \mathcal{H}_\gamma \,, \quad \varphi \otimes \xi \mapsto (\xi_\gamma \varphi)_{\gamma\in\Gamma}$$

it is straightforward to compute the matrix form of the projections $R_i = P_i \otimes Q_i$:

$$(\widehat{R_i})_{\gamma'\gamma} := (W R_i W^*)_{\gamma'\gamma} = \begin{cases} Q_i \, \delta_{\gamma'\gamma} \,, & \gamma', \gamma \in \Gamma_i \\ 0 \;, & \text{otherwise} \,. \end{cases}$$

The commutator of $\widehat{R_i}$ with any $N \in \mathcal{N}$ is

$$\left[\widehat{R_i}\,,\, N\right]_{\gamma'\gamma} = \begin{cases} \left[Q_i\,,\, N_{\gamma'\gamma}\right] \;, & \gamma', \gamma \in \Gamma_i \\ Q_i \, N_{\gamma'\gamma} \;, & \gamma \notin \Gamma_i \,,\; \gamma' \in \Gamma_i \\ -N_{\gamma'\gamma} \, Q_i \;, & \gamma \in \Gamma_i \,,\; \gamma' \notin \Gamma_i \\ 0 \;, & \gamma \notin \Gamma_i \,,\; \gamma' \notin \Gamma_i \,. \end{cases}$$

Step 2: we will check first the Følner condition on the product of generating elements $\pi(M)\,U(\gamma_0)$, $\gamma_0 \in \Gamma$, $M \in \mathcal{M}$ (cf. Eqs. (3.2) and (3.3)). The corresponding matrix elements are given according to Eq. (3.7) by

$$N_{\gamma'\gamma} = \alpha_{\gamma'}^{-1}(M)\, \delta_{\gamma',\gamma_0\gamma} \,.$$

Evaluating the commutator with $\widehat{R_i}$ on the basis elements $\{e_j \, \mathfrak{f}_\gamma\}_{j,\gamma}$, with $\mathfrak{f}_\gamma(\gamma') := \delta_{\gamma\gamma'}$, we get

$$\left[\widehat{R_i}\,,\, (\pi(M) U(\gamma_0))\right] \, e_j \, \mathfrak{f}_\gamma = \begin{cases} \left[Q_i\,,\, \alpha_{\gamma_0\gamma}^{-1}(M)\right] \, e_j \, \mathfrak{f}_{\gamma_0\gamma} \,, & \gamma \in (\gamma_0^{-1}\Gamma_i) \cap \Gamma_i \\ Q_i \, \alpha_{\gamma_0\gamma}^{-1}(M) \, e_j \, \mathfrak{f}_{\gamma_0\gamma} \;, & \gamma \in (\gamma_0^{-1}\Gamma_i) \setminus \Gamma_i \\ -\alpha_{\gamma_0\gamma}^{-1}(M) \, Q_i \, e_j \, \mathfrak{f}_{\gamma_0\gamma} \;, & \gamma \in \Gamma_i \setminus (\gamma_0^{-1}\Gamma_i) \\ 0 \;, & \gamma \notin \Gamma_i \,,\; \gamma \notin \gamma_0^{-1}\Gamma_i \,. \end{cases} \tag{3.10}$$

From this we obtain the following estimates in the Hilbert-Schmidt norm:

$$
\begin{aligned}
&\left\| \left[\widehat{R_i}\,,\, \pi(M)U(\gamma_0) \right] \right\|_2^2 \\
&= \sum_{j,\gamma} \left\| \left[\widehat{R_i}\,,\, (\pi(M)U(\gamma_0)) \right]\; e_j\, \mathfrak{f}_\gamma \right\|^2 \\
&\leq \sum_{\gamma \in (\gamma_0^{-1}\Gamma_i)\cap\Gamma_i} \left\| \left[Q_i\,,\, \alpha^{-1}_{\gamma_0\gamma}(M) \right] \right\|_2^2 + 2|(\gamma_0^{-1}\Gamma_i)\triangle\Gamma_i|\, \|M\|^2\, \|Q_i\|_2^2 \\
&\leq |\Gamma_i| \max_{\gamma\in\Gamma_i} \left\| \left[Q_i\,,\, \alpha^{-1}_{\gamma}(M) \right] \right\|_2^2 + 2|(\gamma_0^{-1}\Gamma_i)\triangle\Gamma_i|\, \|M\|^2\, \|Q_i\|_2^2\;.
\end{aligned}
$$

Using now the hypothesis (3.8) as well as the amenability of Γ via Eq. (2.1) we get finally

$$
\frac{\left\| \left[\widehat{R_i}\,,\, \pi(M)U(\gamma_0) \right] \right\|_2^2}{\left\| \widehat{R_i} \right\|_2^2} \leq \max_{\gamma\in\Gamma_i} \frac{\left\| \left[Q_i\,,\, \alpha^{-1}_{\gamma}(M) \right] \right\|_2^2}{\|Q_i\|_2^2} + 2\,\|M\|^2\, \frac{|(\gamma_0^{-1}\Gamma_i)\triangle\Gamma_i|}{|\Gamma_i|}\;,
$$

and

$$
\lim_i \frac{\left\| \left[\widehat{R_i}\,,\, \pi(M)U(\gamma_0) \right] \right\|_2^2}{\left\| \widehat{R_i} \right\|_2^2} = 0\;, \quad M\in\mathcal{M}\;,\; \gamma_0\in\Gamma\;.
$$

Thus, we have shown the Følner condition for the net $\{R_i\}_i$ on the product of generating elements of the crossed product. By Lemma 2.2 we have that $\{R_i\}_i$ is also a Følner net for their C*-closure

$$
\mathcal{N}_{C^*} := C^*\Big(\{U(\gamma_0), \pi(M) \mid \gamma_0\in\Gamma\,, M\in\mathcal{M}\} \Big)\,.
$$

Step 3: finally, we need to extend the verification of the Følner condition on the whole crossed product von Neumann algebra, that means on the weak closure. Recall from Remark (3.1) that any $N\in\mathcal{N}$ is of the form $N_{\gamma'\gamma} = \alpha_\gamma^{-1}\left(A(\gamma'\gamma^{-1})\right)$ for some $A\colon\Gamma\to\mathcal{M}$. For $N\in\mathcal{N}$ fixed and $\varepsilon>0$ consider a function $\widetilde{A}\colon\Gamma\to\mathcal{M}$ with finite support such that

$$
\sum_{\gamma\in\Gamma} \left\| (A-\widetilde{A})(\gamma) \right\|^2 < \frac{\varepsilon^2}{4}\;. \tag{3.11}
$$

The element $\widetilde{N}$ defined in terms of $\widetilde{A}$ by means of

$$
\widetilde{N}_{\gamma'\gamma} = \alpha_\gamma^{-1}\left(\widetilde{A}(\gamma'\gamma^{-1})\right)
$$

is contained in the C*-closure $\mathcal{N}_{C^*}$. To show the Følner property we will use the second characterization given in Definition 2.1. For this we compute first the corresponding matrix elements:

$$
\left((\mathbb{1}-\widehat{R_i})\, N\, \widehat{R_i} \right)_{\gamma'\gamma} = \begin{cases} (\mathbb{1}-Q_i)\, N_{\gamma'\gamma}\, Q_i\;, & \gamma\,,\,\gamma'\in\Gamma_i \\ N_{\gamma'\gamma}\, Q_i\;, & \gamma\in\Gamma_i\,,\gamma'\in\Gamma_i^c \\ 0\;, & \gamma\in\Gamma_i^c \end{cases}\;,
$$

where Γ_i^c denotes the complement of Γ_i in Γ. From this we have

$$\begin{aligned}
\|(\mathbb{1}-\widehat{R_i})\,(N-\widetilde{N})\,\widehat{R_i}\|_2^2 &= \sum_{\gamma,\gamma'\in\Gamma_i} \|(\mathbb{1}-Q_i)\,(N-\widetilde{N})_{\gamma'\gamma}\,Q_i\|_2^2 \\
&\quad + \sum_{\substack{\gamma\,\in\,\Gamma_i,\\ \gamma'\,\in\,\Gamma_i^c}} \|(N-\widetilde{N})_{\gamma'\gamma}\,Q_i\|_2^2 \\
&\leq \sum_{\substack{\gamma\,\in\,\Gamma_i,\\ \gamma'\,\in\,\Gamma}} \left\|\alpha_\gamma^{-1}\left((A-\widetilde{A})(\gamma'\gamma^{-1})\right)\,Q_i\right\|_2^2 \\
&\leq |\Gamma_i|\,\|Q_i\|_2^2 \sum_{\gamma\in\Gamma} \left\|(A-\widetilde{A})(\gamma)\right\|^2 < |\Gamma_i|\,\|Q_n\|_2^2\;\frac{\varepsilon^2}{4}\,.
\end{aligned}$$

Since the element $\widetilde{N}$ constructed above is contained in the C*-algebra $\mathcal{N}_{C^*}$ we may choose i_0 so that

$$\frac{\|(\mathbb{1}-\widehat{R_i})\,\widetilde{N}\,\widehat{R_i}\|_2}{\|\widehat{R_i}\|_2} < \frac{\varepsilon}{2}\,, \quad \text{whenever } \; i \succcurlyeq i_0.$$

Then, we have

$$\begin{aligned}
\frac{\|(\mathbb{1}-\widehat{R_i})\,N\,\widehat{R_i}\|_2}{\|\widehat{R_i}\|_2} &\leq \frac{\|(\mathbb{1}-\widehat{R_i})\,(N-\widetilde{N})\,\widehat{R_i}\|_2}{\|\widehat{R_i}\|_2} + \frac{\|(\mathbb{1}-\widehat{R_i})\,\widetilde{N}\,\widehat{R_i}\|_2}{\|\widehat{R_i}\|_2} \\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon\,, \quad \text{whenever } \; i \succcurlyeq i_0
\end{aligned}$$

and the proof is concluded. □

The preceding result extends Theorem 3.2 (proved by Bédos), since in the special case where $\mathcal{H}$ is one-dimensional and $\mathcal{M} \cong \mathbb{C}\mathbb{1}$, the crossed product reduces to the group von Neumann algebra $\mathcal{N}_\Gamma$. Moreover, we also have the following immediate consequence for the corresponding concrete C*-crossed product. Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a concrete C*-algebra, Γ a discrete group acting on $\mathcal{A}$ by means of $\alpha\colon \Gamma \to \mathrm{Aut}\mathcal{A}$. Using the definitions in eqs. (3.2) and (3.3) we consider the following representation of the C*-crossed product:

$$\mathcal{A} \rtimes_\alpha \Gamma := C^*\Big(\,\{\pi(A)\mid A\in\mathcal{A}\}\cup\{U(\gamma)\mid\gamma\in\Gamma\}\,\Big) \subset \mathcal{L}(\mathcal{K})\,, \quad \mathcal{K}=\ell^2(\Gamma)\otimes\mathcal{H}\,.$$

Steps 1 and 2 in the previous proof give:

Corollary 3.4. *Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a concrete C*-algebra with Følner net $\{Q_i\}_i$, Γ an amenable, discrete group acting on $\mathcal{A}$ as above $\alpha\colon \Gamma \to \mathrm{Aut}\mathcal{A}$. Let $\{P_i\}_i$ be the net of projection associated to the Følner net $\{\Gamma_i\}$ for Γ. Assume that the action of Γ on $\mathcal{A}$ satisfies:*

$$\lim_i \left(\max_{\gamma\in\Gamma_i} \frac{\|\,\big[Q_i, \alpha_\gamma^{-1}(A)\big]\,\|_2}{\|Q_i\|_2} \right) = 0\,, \quad \text{for all } \; A\in\mathcal{A}\,. \tag{3.12}$$

Then the net $\{R_i\}_i$ with

$$R_i := P_i \otimes Q_i$$

is a Følner net for the C-crossed product $\mathcal{A} \rtimes_\alpha \Gamma$, i.e. the four equivalent conditions in Definition 2.1 are satisfied.*

Proof. See steps 1 and 2 in the proof of Theorem 3.2. □

Remark 3.5. *The compatibility condition (3.8) in the choices of the two Følner nets requires some comments:*

(i) *Note that the compatibility condition already implies that the net $\{Q_i\}_i$ must be a Følner net for the von Neumann algebra $\mathcal{M}$. We have assumed in Theorem 3.3 that $\mathcal{M}$ has a Følner net .*

(ii) *Eq. (3.8) is trivially satisfied in some cases: If Γ is finite, then the compatibility condition is a consequence of the Følner condtion in Eq. (2.2) with $p = 2$.*

Another example is given by the crossed product $\ell^\infty(\Gamma) \rtimes_\alpha \Gamma$, where Γ is a discrete amenable group, $\ell^\infty(\Gamma)$ is the von Neumann algebra acting on the Hilbert space $\ell^2(\Gamma)$ by multiplication and the action α of Γ on $\ell^\infty(\Gamma)$ is given by left translation of the argument. If $\{\Gamma_i\}_i$ is a Følner net for Γ and we denote by $\{P_i\}_i$ the net of finite rank orthogonal projections from $\ell^2(\Gamma)$ onto $\ell^2(\Gamma_i)$, then it is easy to check

$$[P_i, g] = 0 \ , \quad g \in \ell^\infty(\Gamma) \ .$$

Therefore, we have

$$\sup_{\gamma\in\Gamma} \| \left[P_i, \alpha_\gamma^{-1}(g)\right] \|_2 = 0 \ , \quad g \in \ell^\infty(\Gamma)$$

and we may apply Theorem 3.3 to this situation. This particular example is essentially the context in which Bédos studies crossed products in Section 3 of [6]. *In fact, in this particular context one can trace back the existence of a Følner net for the crossed product to the amenability of the discrete group (see Proposition 14 in* [6]*).*

3.2. **The example of the rotation algebra.** The rotation algebra $\mathcal{A}_\theta$, $\theta \in \mathbb{R}$, is the universal C*-algebra generated by two unitaries $\mathbb{U}$ and $\mathbb{V}$ that statisfy the equation

$$\mathbb{U}\mathbb{V} = e^{2\pi i\theta}\, \mathbb{V}\mathbb{U} \ .$$

This is one of the fundamental examples in the theory C*-algebras and has been extensively applied in mathematical physics. Interesting examples from the spectral point of view, like the Mathieu operator or discrete Schrödinger operators with magnetic potentials, can be identified as elements of the rotation algebra. This algebra can also be expressed as a crossed product

$$\mathcal{A}_\theta \cong C(\mathbb{T}) \rtimes_\alpha \mathbb{Z} \ \ ,$$

where $C(\mathbb{T})$ are the continuous function on the unit circle and the action $\alpha\colon \mathbb{Z} \to C(\mathbb{T})$ is given by

$$\alpha_k(f)(z) := f\left(e^{2\pi ik\theta}\, z\right) \ , \ \ f \in C(\mathbb{T}) \ , \ \ z \in \mathbb{T} \ . \tag{3.13}$$

We will apply our main result to the von Neumann crossed product $\mathcal{N}_\theta \cong L^\infty(\mathbb{T}) \otimes_\alpha \mathbb{Z}$. Let $\{\epsilon_k(z) := z^k \mid k \in \mathbb{Z}\}$ be an orhtonormal basis of Hilbert space $\mathcal{H} := L^2(\mathbb{T})$ with the normalized Haar measure. We choose (as in [6, p. 216]) a Følner sequence $\{Q_n\}_{n\in\mathbb{N}_0}$, where Q_n denotes the orthogonal projection onto the subspace generated by $\{\epsilon_i \mid i = 0, \ldots n\}$. We take for the group $\Gamma = \mathbb{Z}$ the Følner sequence $\Gamma_n := \{-n, -(n-1), \ldots, (n-1), n\}$ and denote by P_n the corresponding orthogonal projections on $\ell^2(\mathbb{Z})$. First we verify that the compatibility condition (3.8) for our choice of Følner sequences:

Lemma 3.6. *Consider the previous Følner sequence $\{Q_n\}_n$ for the von Neumann algebra $\mathcal{M} := L^\infty(\mathbb{T})$ and the group action $\alpha\colon \mathbb{Z} \to L^\infty(\mathbb{T})$ definded in Eq. (3.13). Then for $g \in L^\infty(\mathbb{T})$ we have*

$$\| \left[Q_n, \alpha_k^{-1}(g)\right] \|_2 = \| \left[Q_n, g\right] \|_2 \ , \quad k \in \mathbb{Z} \ ,$$

and

$$\lim_{n\to\infty} \left(\max_{k\in\Gamma_n} \frac{\| \left[Q_i, \alpha_k^{-1}(g)\right] \|_2}{\|Q_i\|_2} \right) = 0 \ , \quad \text{for all } \ g \in L^\infty(\mathbb{T}) \ .$$

Proof. The first equation is a consequence of the some elementary statements in harmonic analysis:

$$
\begin{aligned}
\| \left[Q_n, \alpha_k^{-1}(g)\right] \|_2^2 &= \sum_{l=-\infty}^{\infty} \|(Q_n\, \alpha_{-k}(g) - \alpha_{-k}(g)\, Q_n)\, \epsilon_l\|^2 \\
&= \sum_{l=-n}^{n} \|(\mathbb{1} - Q_n)\, \alpha_{-k}(g)\, \epsilon_l\|^2 \;+ \sum_{l\in(\mathbb{Z}\setminus\{-n,\ldots,n\})} \|Q_n\, \alpha_{-k}(g)\, \epsilon_l\|^2 \\
&= \sum_{m\in(\mathbb{Z}\setminus\{-n,\ldots,n\})} \; \sum_{l=-n}^{n} \left| e^{i2\pi k\theta(m-l)}\, \widehat{g}(m-l)\right|^2 \\
&\quad \sum_{m=-n}^{n} \; \sum_{l\in(\mathbb{Z}\setminus\{-n,\ldots,n\})} \left| e^{i2\pi k\theta(m-l)}\, \widehat{g}(m-l)\right|^2 \\
&= \| \left[Q_n, g\right] \|_2^2 \;.
\end{aligned}
$$

The second equation follows directly from the first equation and the fact that $\{Q_n\}_n$ is a Følner sequence for $L^\infty(\mathbb{T})$. □

Proposition 3.7. *Let $\mathcal{N}_\theta \cong L^\infty(\mathbb{T}) \otimes_\alpha \mathbb{Z}$, with θ irrational, be the von Neumann algebra associated to the rotation algebra and acting on $\mathcal{K} = \ell^2(\mathbb{Z}) \otimes \mathcal{H}$.*

(i) *Consider the sequences $\{Q_n\}_{n\in\mathbb{N}_0}$ and $\{P_n\}_{n\in\mathbb{N}_0}$ defined before. Then*

$$\{R_n := P_n \otimes Q_n\}_{n\in\mathbb{N}_0}$$

is a Følner sequence for $\mathcal{N}_\theta$.

(ii) *Let $T \in \mathcal{N}_\theta$ be a selfadjoint element in the rotation algebra and denote by μ_T the spectral measure associated with the unique trace of $\mathcal{N}_\theta$. Consider the compressions $T_n := R_n T R_n$ and denote by μ_T^n the probability measures on $\mathbb{R}$ supported on the spectrum of (T_n). Then $\mu_T^n \to \mu_T$ weakly, i.e.*

$$\lim_{n\to\infty} \frac{1}{d_n}\Big(f(\lambda_{1,n}) + \cdots + f(\lambda_{d_n,n})\Big) = \int f(\lambda)\, d\mu(\lambda) \;, \quad f \in C_0(\mathbb{R}) \;,$$

where d_n is the dimension of the R_n and $\{\lambda_{1,n}, \ldots, \lambda_{d_n,n}\}$ are the eigenvalues (repeated according to multiplicity) of T_n.

Proof. Part (i) follows from Theorem 3.3 and Lemma 3.2. To prove Part (ii) recall $\mathcal{N}_\theta$ is a factor of type II_1 and therefore has a unique trace. The rest of the statement is a direct application of Theorem 6 (iii) in [6] to the example of the rotation algebra. □

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